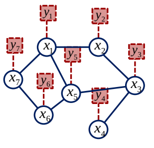
Generalization of Concave and Convex Decomposition in Kikuchi Free Energy

Department of Computational Intelligence
and Systems Science,
Tokyo Institute of Technology, Japan
Yu Nishiyama, Sumio Watanabe

Contents

Background



- Graphical models
  - represent statistical dependencies between random variable nodes.
- Inference
  - to compute marginals of single hidden nodes or subsets of hidden nodes, given observed nodes.
- Message passing algorithms
  - Belief propagation
- Many applications
  - Bayesian networks, Error-correcting codes, CDMA, image processing.

Background

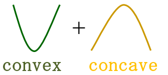
- BP fixed-points = Stationary points of Bethe free energy
- Kikuchi free energy is an extension of Bethe free energy.
- CCCP (A.L. Yuille, 2001)

	Convergence	Computational cost
Belief propagation (BP)	Not Guaranteed	Moderate
Concave convex Procedure(CCCP)	Guaranteed	High

CCCP

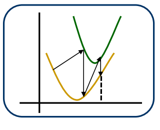
- Objective functional

$$F(\mathbf{r}) \equiv F_{vex}(\mathbf{r}) + F_{cave}(\mathbf{r})$$



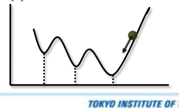
- Recursive equation

$$\nabla F_{vex}(\mathbf{r}^{(t+1)}) = -\nabla F_{cave}(\mathbf{r}^{(t)})$$



- It follows that

$$F(\mathbf{r}^{(t+1)}) \le F(\mathbf{r}^{(t)})$$



Bethe and Kikuchi free energies

- Bethe free energy

$$F_{Bethe}(\{r_i\},\{r_{ij}\}) = S_{Bethe} + \sum_{\{ij\}\in B} < E_{ij}(x_i, x_j) >_{r_{ij}}$$

$$= \underbrace{-\sum_{\{ij\}\in B} S_{ij} - \sum_{i\in V} S_i}_{\text{convex}} + \underbrace{\sum_{i\in V} |N_i| S_i}_{\text{concave}} + \underbrace{\sum_{\{ij\}\in B} < E_{ij}(x_i, x_j) >_{r_{ij}}}_{\text{linear}}$$

- Kikuchi free energy

$$F_{Kikuchi}(\{r_\alpha\}) = S_{kikuchi} + \sum_{\alpha\in R} c_\alpha < E_\alpha(\mathbf{x}_\alpha) >_{r_\alpha}$$

$$= \underbrace{-\sum_{\substack{\alpha\in R\\ c_\alpha>0}} c_\alpha S_\alpha(r_\alpha)}_{\text{convex}} \underbrace{- \sum_{\substack{\alpha\in R\\ c_\alpha<0}} c_\alpha S_\alpha(r_\alpha)}_{\text{concave}} + \underbrace{\sum_{\alpha\in R} c_\alpha < E_\alpha(\mathbf{x}_\alpha) >_{r_\alpha}}_{\text{linear}}$$

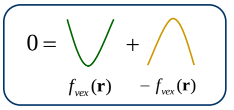
→ CCCP iterative algorithm

NCCCP for minimizing Bethe and Kikuchi free energies

- feature
  - includes CCCP for Bethe and Kikuchi.
  - is guaranteed to monotonically decrease Bethe and Kikuchi free energies.
  - can reduce huge computational cost.
  - more stable.

NCCCP for Kikuchi (Key Idea)

- Trivial Pair Creation

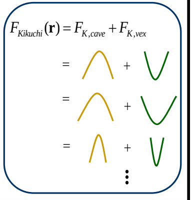


- Proposal of NCCCP

$$F_{K,vex}(\mathbf{r}) = F_{Kikuchi}(\mathbf{r}) + f_{vex}(\mathbf{r})$$
$$F_{K,cave}(\mathbf{r}) = -f_{vex}(\mathbf{r})$$

  - Specially

$$f_{vex}(\mathbf{r},\mathbf{u}) = -\sum_{\alpha\in R} u_{S,\alpha} S_\alpha(r_\alpha) + \sum_{\alpha\in R} u_{E,\alpha} \langle E_\alpha \rangle_{r_\alpha}$$



Decomposition with parameters

- Convex part

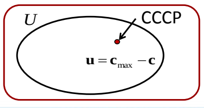
$$F_{K,cave}(\mathbf{r},\mathbf{u}) = \sum_{\alpha\in R} u_{S,\alpha} S_\alpha(r_\alpha) - \sum_{\alpha\in R} u_{E,\alpha} \langle E_\alpha(\mathbf{x}_\alpha)\rangle_{r_\alpha}$$

- Concave part

$$F_{K,vex}(\mathbf{r},\mathbf{u}) = -\sum_{\alpha\in R} (c_\alpha + u_{S,\alpha}) S_\alpha(r_\alpha) + \sum_{\alpha\in R} (c_\alpha + u_{E,\alpha}) \langle E_\alpha(\mathbf{x}_\alpha)\rangle_{r_\alpha}$$

- Feasible set U

$$U = \{\mathbf{u} \in \mathbb{R}^{|R|} \mid u_\alpha > \max\{0, -c_\alpha\}, {}^\forall \alpha \in R\}$$



NCCCP algorithm for Kikuchi

- Algorithm (NCCCP)
  - Outer loop

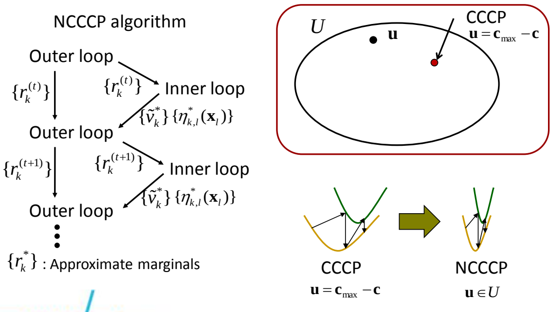
$$r_k^{(t+1)} \leftarrow \{r_k^{(t)}\}^{\frac{u_k}{c_k+u_k}} \exp[-\tilde{v}_k - \sum_{\beta\in sub_d(k)} \frac{\eta_{k,\beta}(\mathbf{x}_\beta)}{c_k+u_k} + \sum_{\gamma\in \sup_d(k)} \frac{\eta_{\gamma,k}(\mathbf{x}_k)}{c_k+u_k} - \frac{c_k E_k}{c_k+u_k}] \quad k\in R$$

  - Inner loop

$$\exp(\tilde{v}_k^{(\tau+1)}) \leftarrow \sum_{\mathbf{x}_k} \{r_k^{(t)}\}^{\frac{u_k}{c_k+u_k}} \exp[-\sum_{\beta\in sub_d(k)} \frac{\eta_{k,\beta}^{(\tau)}(\mathbf{x}_\beta)}{c_k+u_k} + \sum_{\gamma\in \sup_d(k)} \frac{\eta_{\gamma,k}^{(\tau)}(\mathbf{x}_k)}{c_k+u_k} - \frac{c_k E_k}{c_k+u_k}]$$

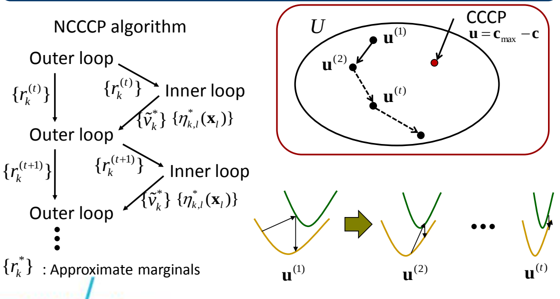
$$\exp\{(\frac{1}{c_k+u_k} + \frac{1}{c_l+u_l})\eta_{k,l}^{(\tau+1)}(\mathbf{x}_l)\} \leftarrow \sum_{\mathbf{x}_{k\backslash l}} \frac{\{r_k^{(t)}\}^{\frac{u_k}{c_k+u_k}} \exp[-\tilde{v}_k^{(\tau)} - \sum_{\beta\in sub_d(k)\backslash l} \frac{\eta_{k,\beta}^{(\tau)}(\mathbf{x}_\beta)}{c_k+u_k} + \sum_{\gamma\in \sup_d(k)} \frac{\eta_{\gamma,k}^{(\tau)}(\mathbf{x}_k)}{c_k+u_k} - \frac{c_k E_k}{c_k+u_k}]}{\{r_l^{(t)}\}^{\frac{u_l}{c_l+u_l}} \exp[-\tilde{v}_l^{(\tau)} - \sum_{\beta\in sub_d(l)} \frac{\eta_{l,\beta}^{(\tau)}(\mathbf{x}_\beta)}{c_l+u_l} + \sum_{\gamma\in \sup_d(l)\backslash k} \frac{\eta_{\gamma,l}^{(\tau)}(\mathbf{x}_l)}{c_l+u_l} - \frac{c_l E_l}{c_l+u_l}]}$$

$$k \in R \quad l \in sub_d(k)$$



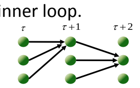
Remark

Even if free vector $\mathbf{u}$ is changed within U at every outer loop, NCCCP is guaranteed to monotonically decrease Kikuchi free energy.

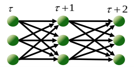


Example (Gaussians)

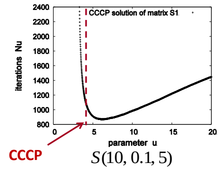
- Bethe free energy
- Free vector: constant
- Two different iterative methods w.r.t. inner loop.
  - Asynchronous
    - Update *one* parameter at every inner loop.



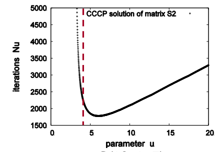
  - Synchronous
    - Update *all* parameters at every inner loop.



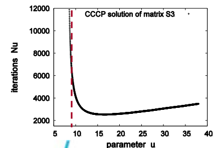
Numerical Results (Asynchronous)



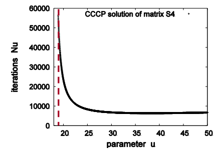
$S(10, 0.1, 5)$



$S(10, 2, 5)$

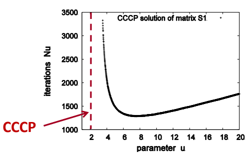


$S(10, 0.1, 10)$

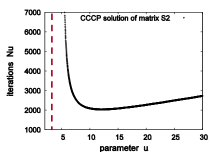


$S(10, 0.1, 20)$

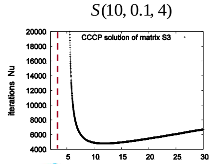
Numerical Results (Synchronous)



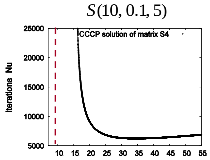
$S(10, 0.1, 4)$



$S(10, 0.1, 5)$



$S(10, 2, 5)$



$S(10, 0.1, 10)$

Summary

- NCCCP algorithm for Kikuchi is presented.
  - more general than conventional CCCP.
  - general concave and convex decomposition.
  - Reduce expensive computational cost.
  - more stable not depending on synchronous or asynchronous.